Pergamon

Int. J. Solids Structures Vol. 35, No. 33, pp. 4369–4379, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00239–4

ON THE BRACING OF STRUCTURES, AND THE SPECIAL THEORY OF RELATIVITY

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(*Received* 19 *July* 1996; *in revised form* 29 *July* 1997)

Abstract—Stiffening a structure with a brace generally changes its buckling loads. Bracing to a given load with a brace of known geometry leads to a simple eigenvalue problem, but if the geometric connections are left open, an investigation of the possibilities leads to a description in Minkowski space—the same as is used in interpreting the special theory of relativity. Despite clear differences, the two problems have surprising parallels. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Bracing is adding stiffness to a structure, usually to increase the buckling load or the vibration frequency. It often takes the form of added members, and can be classified according to the number of ways this added substructure can be stressed, giving the rank of the brace. Rank 1 bracing is the simplest (an added truss member is a typical example), but is also quite general, in that bracing of any rank can be constructed from separately applied rank 1 braces. It is well known that a rank 1 brace can increase the fundamental buckling load of a structure up to, but not beyond, the second buckling load of the unbraced structure.

As an example, consider the cantilever column, modelled with two finite elements, shown in Fig. 1. Unbraced, this structure has a fundamental buckling load P_1 (strictly P_1L^2/EI) of 2.469 and a second load P_2 of 22.946 (approximations to $\pi^2/4$ and $9\pi^2/4$ in an exact model). A spring prop placed at the free end increases both of these loads, but even as the brace stiffness k (strictly kL^3/EI) approaches infinity and fully supports the tip, the fundamental load is increased to 20.709, less than the second load of the original structure.

The original problem has eigenvalue equations

$$\mathbf{K}(P)\mathbf{u} = \mathbf{0} \tag{1}$$

where the symmetric stiffness matrix $\mathbf{K}(P)$ depends on the loading, here represented by a single parameter P. Solutions of eqn (1) are associated pairs $(P_i, \mathbf{u}_i)$ which give a buckling load and a buckled shape or mode.

On adding a brace with geometric connection $\mathbf{g}$ and stiffness k, a new structure is described by stiffness equations

$$(\mathbf{K}(P) + \mathbf{g}^{\mathrm{T}}k\mathbf{g})\mathbf{u} = \mathbf{0} \tag{2}$$

but can also be written in a mixed stiffness and flexibility form

$$\begin{bmatrix} \mathbf{K}(P) & \mathbf{g}^{\mathrm{T}} \\ \mathbf{g} & -f \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ N_{\mathrm{b}} \end{bmatrix} = \mathbf{0} \tag{3}$$

where f is the brace flexibility $(=1/k)$, and N_{b} is the force in the brace after buckling. A rank 1 brace with connection $\mathbf{g}$ will brace a single displacement $u_{\mathrm{b}} = \mathbf{gu}$. Freedom u_1 is the only one braced in the above example, and the connection for this is $\mathbf{g} = [1\ 0\ 0\ 0]$.

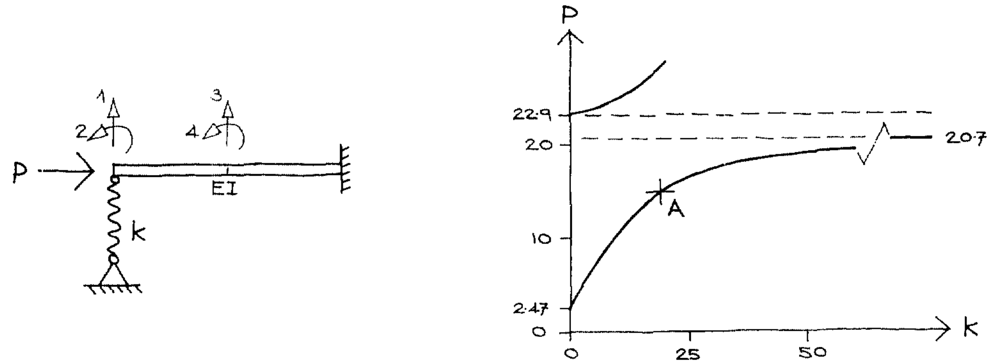

Fig. 1. Bracing of a cantilever.

The mixed formulation leads to a 1×1 eigenvalue problem, with solution (Lawther, 1995)

$$-1/k = -f = \mathbf{g}\mathbf{K}^{-1}\mathbf{g}^{T}. \tag{4}$$

Equation (4) allows us to choose a load P and then calculate the bracing stiffness needed to produce this as the buckling load. Choosing $P = 15$ as the target buckling load of the propped cantilever, the stiffness of the original, unpropped cantilever is formulated at this load, and with $\mathbf{g} = [1\ 0\ 0\ 0]$, eqn (4) gives $k = 19.26$. This is the brace stiffness needed to raise the fundamental buckling load to 15, shown as point A in Fig. 1. Similarly investigating at $P = 21$ gives $k = -137.27$, with the negative stiffness indicating it is physically impossible to prop a cantilever to a fundamental buckling load of 21. In this way we can readily decide if a chosen brace will work. An obvious extension is to consider which other braces will work.

Having chosen a load, eqn (4) allows us to classify a given brace geometry $\mathbf{g}$ as *feasible* or *infeasible* according to whether $\mathbf{g}\mathbf{K}^{-1}\mathbf{g}^{T} \leqslant 0$ or >0. Braces with $\mathbf{g}\mathbf{K}^{-1}\mathbf{g}^{T} = 0$ are feasible, but require infinite brace stiffness k, and comprise the *feasible boundary*. This paper looks at some of the properties of feasible braces.

2. BACKGROUND

Feasibility of a brace is a function of the structure and the load P to be achieved, and the connection geometry $\mathbf{g}$. The magnitude of $\mathbf{g}$ is immaterial to feasibility, and is normed to $\mathbf{g}\mathbf{g}^{T} = 1$. The load P is restricted to vary between the fundamental buckling load P_1 and the second load P_2, the maximum load that can be achieved with a single brace.

The eigenvalue equations are assumed to be of the algebraic form typical of finite formulations;[1]

$$\mathbf{K}(P)\mathbf{u} = (\mathbf{K} - P\mathbf{S})\mathbf{u} = \mathbf{0}. \tag{1a}$$

For any non-singular matrix $\mathbf{A}$

$$\mathbf{g}\mathbf{K}(P)^{-1}\mathbf{g}^{T} = \mathbf{g}\mathbf{A}\mathbf{A}^{-1}\mathbf{K}(P)^{-1}\mathbf{A}^{-T}\mathbf{A}^{T}\mathbf{g}^{T} = \mathbf{h}\bar{\mathbf{K}}(P)^{-1}\mathbf{h}^{T}, \tag{5}$$

which is a transformation of coordinates $\mathbf{h} = \mathbf{g}\mathbf{A}$, implying $\bar{\mathbf{K}} = \mathbf{A}^{T}\mathbf{K}\mathbf{A}$ and $\mathbf{K} = \mathbf{A}^{-T}\bar{\mathbf{K}}\mathbf{A}^{-1}$. $\mathbf{K}$ is singular when $P = P_1$ and $P = P_2$, and is nonsingular for all values in between. For all P between P_1 and P_2, both $\mathbf{A}^{-1}$ and $\bar{\mathbf{K}}$ are non singular, but as $P \to P_1$ or P_2, one (or perhaps both) will become singular.

[1] Derived results are likely to apply for transcendental formulations of $\mathbf{K}$ as well, but this is not guaranteed as the signature of $\mathbf{K}$ (see later in this section) may not be a monotonic function of P in the range considered.

Transformations that create $\bar{\mathbf{K}}$ as a diagonal matrix are of special interest. Gauss factorisation is one, giving the familiar

$$\mathbf{K} = \mathbf{LDL}^{\mathrm{T}}. \tag{6}$$

Another, particularly important diagonalising transformation uses the matrix of eigenvectors. Writing $\mathbf{U}$ as the matrix with the eigenvectors $\mathbf{u}_i$ as its columns, the transformation $\boldsymbol{\alpha} = \mathbf{gU}$ gives

$$\mathbf{gK}(P)^{-1}\mathbf{g}^{\mathrm{T}} = \boldsymbol{\alpha}\mathbf{U}^{-1}(\mathbf{K}-P\mathbf{S})^{-1}\mathbf{U}^{-\mathrm{T}}\boldsymbol{\alpha}^{\mathrm{T}} = \boldsymbol{\alpha}(\mathbf{I}-P\mathbf{D}^{\mathrm{S}})^{-1}\boldsymbol{\alpha}^{\mathrm{T}}. \tag{7}$$

Just as $\mathbf{g}$ is the connection of the brace to the freedoms, $\boldsymbol{\alpha}$ gives the connection to the modes. $\mathbf{U}$ is a matrix independent of load, and simultaneously diagonalises both $\mathbf{K}$ and $\mathbf{S}$. It is unique to within the scale and order of the columns, and is the only matrix to give the simultaneous diagonalisation. Here the scale is defined by transformation of the linear stiffness, $\mathbf{K}$, to $\mathbf{I}$, and the terms of $D^{\mathrm{S}}_{\mathrm{ii}}$ are chosen as the positive values in order of decreasing magnitude, followed by the negative in like order, then the zero diagonal terms. With this definition α_1 is the connection to the fundamental mode, α_2 is to the second mode, α_3 to the third, and so on (the details of modes after the third are of very little interest here).[2] The buckling loads are $P_i = 1/D^{\mathrm{S}}_{\mathrm{ii}}$, with some minor complication of $D^{\mathrm{S}}_{\mathrm{ii}} = 0$.

Diagonalisations of any symmetric matrix by real non-singular transformations have a property of fundamental importance. The number of positive terms is the same in all cases. So is the number of negative terms, and the number of zero terms (Parlett, 1980; Myškis, 1975). Respectively, these numbers are the positive (π), negative (ν) and zero (ζ) Sylvester inertias of the matrix (ν is also known as the *signature* of the matrix). $\mathbf{K}(P)$ has $\zeta = 0$ except that when P is an eigenvalue, ζ is its multiplicity. In our region of interest, $P_1 < P < P_2$, $\mathbf{K}(P)$ has $\zeta = 0$ and $\nu = 1$, that is, the diagonal form will have exactly one negative term, with the rest positive. With the ordering of $D^{\mathrm{S}}_{\mathrm{ii}}$ given above, the negative term will be the first.

3. ON THE BRACING OF STRUCTURES....

3.1. *In the space of the eigenvectors*

Working in the $\boldsymbol{\alpha}$ space of the eigenvectors, feasible braces have

$$\boldsymbol{\alpha}(\mathbf{I}-P\mathbf{D}^{\mathrm{S}})^{-1}\boldsymbol{\alpha}^{\mathrm{T}} = \Sigma\frac{\alpha_i^2}{1-PD^{\mathrm{S}}_{\mathrm{ii}}} \leqslant 0$$

and with $P_i = 1/D^{\mathrm{S}}_{\mathrm{ii}}$,

$$\Sigma\frac{\alpha_i^2 P_i}{P_i - P} \leqslant 0. \tag{8}$$

If $D^{\mathrm{S}}_{\mathrm{ii}} = 0$ (an *infinite mode*), $(P_i/(P_i - P))$ is taken as 1. When combined with the norm $\boldsymbol{\alpha\alpha}^{\mathrm{T}} = \Sigma\alpha_i^2 = 1$, eqn (8) gives

$$\Sigma\alpha_i^2\left(\frac{P_i}{P_i-P}+\frac{P_1}{P-P_1}\right) \leqslant \frac{P_1}{P-P_1} \tag{9}$$

or

[2] A note on repeated solutions: if P_1 is a repeated solution then rank 1 bracing offers nothing, as there are no loads between P_1 and P_2. A repeated P_2 has relatively minor consequences that are discussed later, and any other repeated solution has no interest. The eigenvectors $\mathbf{u}_i$ used in the definition of $\mathbf{U}$ must now be interpreted as invariant subspaces, but again, this is of no interest except for the repeated P_2.

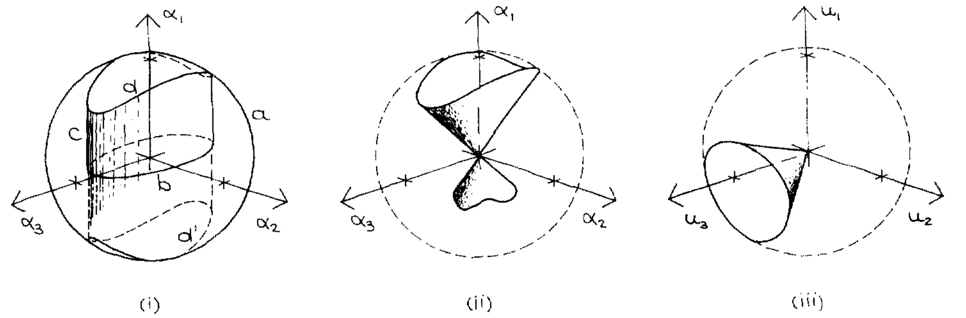

Fig. 2. The feasible region.

$$c_2\alpha_2^2 + c_3\alpha_3^2 + \cdots \leqslant c_0. \tag{9a}$$

For $P_1 < P < P_2$ all coefficients are positive, and c_0 is the smallest. Equation (9) describes a region that is a cylinder, or tube, in the α_1 direction, and is elliptical with principal axes in the other α_i directions.

In Fig. 2(i) the sphere $\boldsymbol{\alpha\alpha}^T = 1$ is shown as a, and b shows the cross section of the tube c. Curves d and d' are the intersections of this cylinder with the sphere, and define the feasible boundary. The feasible region is all vectors on or inside these curves, which is the surface and interior of the double cone of Fig. 2(ii), although only one of the cones need be considered, with regard to symmetry. Transformation back to the original freedoms produces a general conical region, shown in Fig. 2(iii).

Semiaxes r_i of the ellipse shown as curve b in Fig. 2(i) are given by

$$r_i^2 = c_0/c_i = \frac{P_1}{P}\left(\frac{P_i - P}{P_i - P_1}\right) \text{ or } \frac{P_1}{P} \tag{10}$$

with the second expression applying for an infinite mode. All are less than 1. And $\partial r_i^2/\partial P$ is negative for all P_i, whether positive, negative or infinite. The feasible region shrinks as P increases, continuously becoming a proper subset of itself, until as $P \to P_2$ $r_2 \to 0$, and the ellipse becomes fully flattened in the α_2 direction. If a brace is to achieve a load of P_2, it must not connect to the second mode. This conclusion was reached from a less geometric viewpoint in Barbato and Lawther, 1996. The progression of the region of feasible braces with increasing buckling load is shown in Fig. 3. At $P = P_1$ the feasible region is the complete sphere, with the $\alpha_1 = 0$ equator the feasible boundary. P_1 is the fundamental buckling load, and no brace is needed to achieve it, which is any brace whatever, of zero stiffness. In Fig. 3(i) P is slightly $> P_1$ and the feasible region is almost the full space, with just a small region of infeasible braces around the $\alpha_1 = 0$ equator. As P increases the region of feasible braces contracts symmetrically about the α_1 axis, in all directions, through intermediate stages like Fig. 3(ii) until finally Fig. 3(iii) shows the flattened region as $P \to P_2$. The flattened cone cannot occupy the whole of the α_2 plane, and there is a region of

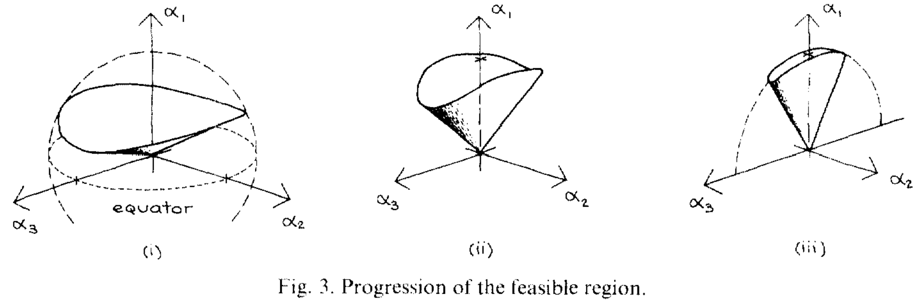

Fig. 3. Progression of the feasible region.

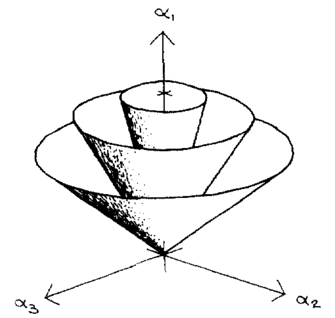

Fig. 4. Coincident second modes.

braces which do not connect to the second mode, but are still infeasible, shown dotted in Fig. 3(iii). This point is discussed further, with examples, in the previous reference.

At $P = P_2$ the semiaxes are

$$r_i^2 = \frac{P_1}{P_2}\left(\frac{P_i - P_2}{P_i - P_1}\right) \quad \left(\text{or } \frac{P_1}{P_2}\right). \tag{10a}$$

This confirms that the ellipse is flat in the α_2 direction, and shows that the radius increases for each successive positive mode, approaching $(P_1/P_2)^{1/2}$. The radii of the negative modes are all greater than this limit, and progressively approach it from above.

The effect of a repeated second mode is seen immediately from eqns 10 and 10a, and is shown in Fig. 4. The ellipse is circular in all directions of the multiplicity of the solution, and contracts uniformly in these directions, becoming flattened in all as $P \to P_2$.

The brace $\boldsymbol{\alpha}_1 = \mathbf{e}_1^{\mathrm{T}} = [1\ 0\ 0\ldots]$, at the centre of the ellipses connects to the first mode only, and is always feasible. In rare cases it is the only feasible brace as $P \to P_2$. And it is the most efficient in the sense that it gives a constant $\partial P/\partial k$. All others give diminishing returns for increased stiffness. In the coordinates of the freedoms this brace is $\mathbf{g}_1 = \boldsymbol{\alpha}_1\mathbf{U}^{-1}$, i.e. the first row of $\mathbf{U}^{-1}$.

The set of feasible braces at any load P is a closed convex set in the space of the eigenvectors. It is therefore a closed convex set in any coordinates under a constant transformation matrix (constant, as in the same for all vectors, not necessarily the same for all P). However the central brace may not be preserved under mappings, and in general $\mathbf{g}_1$ will not appear at the middle of the feasible region, when plotted in $\mathbf{u}$ space and normed to $\mathbf{g}\mathbf{g}^{\mathrm{T}} = 1$. Indeed, this region is unlikely to be symmetrical, and therefore will not have a well-defined middle.

3.2. *Radially symmetric formulations*

The eigenvector space gives a symmetrical description of the cone of feasible braces, and the central brace $\boldsymbol{\alpha} = [1\ 0\ 0\ldots]$ has some unique properties. Other symmetric forms are clearly possible, and the central brace in these may also have a significant role. To investigate this, the problem will first be written in a standardised form of a circular cone with a right angled apex, which can be generated simply from a diagonal form—given $\mathbf{K} = \mathbf{A}^{\mathrm{T}}\mathbf{D}\mathbf{A}$, the rows of $\mathbf{A}$ are scaled to $\bar{\mathbf{A}}$, producing $\mathbf{K} = \bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{I}}\bar{\mathbf{A}}$, where $\bar{\mathbf{I}} = \ulcorner \pm 1 \lrcorner$. Writing $\boldsymbol{\beta} = \mathbf{g}\bar{\mathbf{A}}^{-1}$, the feasible region is now

$$\boldsymbol{\beta}\bar{\mathbf{I}}\boldsymbol{\beta}^{\mathrm{T}} \leqslant 0. \tag{11}$$

$\bar{\mathbf{I}}$ has one element of -1 and the rest are $+1$. For illustration purposes,[3] the negative element is assumed to be the first, and the above equation is

[3] It may not be possible to make the negative element the first. For example, if the Gauss transformation is further transformed in this way, the result is the signed Choleski, where the position of the negative element is determined by the transformation being triangular, and is very unlikely to be the first. This is of no importance for the argument here, but is central to the treatment of eqns (15) and (16) in section 3.3.

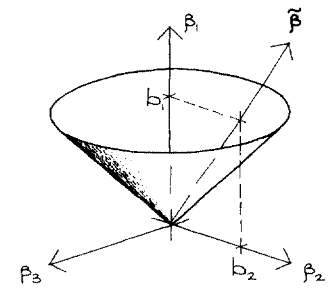

Fig. 5. Radially symmetric formulation.

$$-\beta_1^2+\beta_2^2+\beta_3^2+\cdots\leqslant 0. \tag{11a}$$

Equation (9a) becomes

$$2\beta_2^2+2\beta_3^2+\cdots\leqslant 1 \tag{9b}$$

which is the circular, right angled cone shown in Fig. 5. In this representation the feasible region remains constant, but the coordinates of the brace, $\boldsymbol{\beta}$, change with load. As P increases, the feasible region, given by eqn (9b), or alternatively by $2\beta_1^2 \geqslant 1$, gets progressively 'less dense' as braces of a given $\mathbf{g}$ move out. At the limit of $P = P_2$ the feasible region is 'infinitely sparse' in the mapped direction of $\mathbf{g}_2$ (or $\boldsymbol{\alpha}_2$), with 2 points on the feasible boundary and all points on a connecting line representing the same brace.

To determine any significance attached to the central brace in this radially symmetric formulation, consider 2 such representations

$$-1/k = \mathbf{g}\mathbf{K}^{-1}\mathbf{g}^{\mathrm{T}} = \boldsymbol{\beta}\bar{\mathbf{I}}\boldsymbol{\beta}^{\mathrm{T}} = \gamma\bar{\mathbf{I}}\gamma^{\mathrm{T}} \tag{12}$$

related by $\boldsymbol{\beta} = \gamma\mathbf{A}$, from which

$$\mathbf{A}\bar{\mathbf{I}}\mathbf{A}^{\mathrm{T}} = \bar{\mathbf{I}}. \tag{13a}$$

$\bar{\mathbf{I}}$ has full rank, so $\mathbf{A}$ is also non singular. Right hand multiplication by $\bar{\mathbf{I}}$ leads to the orthonormality of sorts

$$\mathbf{A}^{-1} = \bar{\mathbf{I}}\mathbf{A}^{\mathrm{T}}\bar{\mathbf{I}} \tag{13b}$$

which combines with eqn (13a) to give

$$\mathbf{A}^{\mathrm{T}}\bar{\mathbf{I}}\mathbf{A} = \bar{\mathbf{I}}. \tag{13c}$$

Only braces from the interior of the $\boldsymbol{\beta}$ cone can be mapped onto the central vector of the γ cone. This is obvious physically, but can be established formally by noting that the image of $\gamma = [1\ 0\ 0\ldots]$ in the $\boldsymbol{\beta}$ space under the transformation $\mathbf{A} = [a_{ij}]$ is $\tilde{\boldsymbol{\beta}} = [a_{11}\ a_{12}\ a_{13}\ldots]$. Expanding the 1,1 element of eqn (13a),

$$-a_{11}^2+a_{12}^2+a_{13}^2+\cdots = -1$$

which is <0, and therefore in the interior of the cone of feasible braces.

And any such vector $\tilde{\boldsymbol{\beta}}$ can be mapped onto the central vector of the γ space. $\tilde{\boldsymbol{\beta}}$ is assumed to lie in the β_1, β_2 plane (without loss of generality), with coordinates $[b_1\ b_2\ 0\ 0\ldots]$. The transform

$$\mathbf{A} = \begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix}; \quad c_i = b_i/\sqrt{b_1^2 - b_2^2} \tag{14}$$

achieves the required result, and is real since $b_1^2 - b_2^2 > 0$.

Any brace inside the $\boldsymbol{\beta}$ cone, and only those braces, can be mapped to the centre of the γ cone. There is no privileged position. A feasible brace is either on the feasible boundary or can be considered as in the centre (or for that matter, any other part) of the feasible region.

3.3. *The Choleski description*

The $\boldsymbol{\alpha}$ space of the eigenvectors gives the most useful description of the problem, but is not very attractive computationally as it requires a complete solution of the eigenvalue problem of the unbraced structure. The signed Choleski transformation

$$\mathbf{K}(P) = \bar{\mathbf{L}}\bar{\mathbf{I}}\bar{\mathbf{L}}^{\mathrm{T}} \tag{15a}$$

implying

$$\mathbf{K}(P)^{-1} = \bar{\mathbf{L}}^{-\mathrm{T}}\bar{\mathbf{I}}\bar{\mathbf{L}}^{-1}, \quad \boldsymbol{\beta} = \mathbf{g}\mathbf{L}^{-\mathrm{T}} \text{ and } \mathbf{g} = \boldsymbol{\beta}\mathbf{L}^{\mathrm{T}} \tag{15b,c,d}$$

uses the lower triangular $\bar{\mathbf{L}}$, is radially symmetric, and has significant computational advantages. As discussed in a previous footnote, $\bar{\mathbf{I}}$ must have exactly one diagonal term of -1, but it can appear in any position, p. Braces $\mathbf{g}$ which connect to higher numbered freedoms only, equally do so in the $\boldsymbol{\beta}$ space, and are immediately seen as infeasible. (An alternative physical argument leads to the same conclusion: a factorisation by Crout-Choleski starts with a structure with the single freedom u_1 and sequentially introduces and factorises each subsequent freedom. By the time freedom p has been included the (sub)structure has buckled, with all freedoms > p still fully supported. Releasing these and then applying a brace to them is not going to prevent the buckle.) A brace that is not feasible remains so with increasing P, so the negative can only change position by working its way up the diagonal: $\mathrm{p}(P)$ is a monotonically decreasing function, decrementing at the point where the contracting feasible region no longer includes any point in the subspace of u_{p} and all higher numbered freedoms.

This exclusion of braces only requires that we know $\bar{\mathbf{I}}$, and therefore p. Other single freedom braces can be included (or excluded) through eqns (11) and (15c), which show that, on forming $\bar{\mathbf{L}}^{-\mathrm{T}} = [l_{ij}]$, freedom f is in the feasible region iff

$$-1/k = -l_{\mathrm{fp}}^2 + \sum_{j \neq \mathrm{p}} l_{\mathrm{fj}}^2 \leqslant 0. \tag{16}$$

Further to this, a feasible brace can be generated by choosing any $\boldsymbol{\beta}$ within the feasible cone (eqn (11a)) and calculating $\mathbf{g} = \boldsymbol{\beta}\mathbf{L}^{\mathrm{T}}$. For example, the central brace $\boldsymbol{\beta} = [0\ 0\ 1\ 0\ldots]$ (the 1 appears in position p), which is trivially feasible, maps to the pth row of $\mathbf{L}^{\mathrm{T}}$. Of course, $\mathbf{L}$ is a function of P, and these calculations give results which depend on the target load used.

Convexity admits any other brace with positive combinations of braces already found to be in the feasible region.[4]

3.4. *Bracing the cantilever*

This Choleski formulation is now applied to the introductory cantilever problem, for various target loads.

[4] Calculation will show if a brace lies in the feasible region, but it could be in either cone. Convexity applies to each cone separately. Before combinations are formed we must be sure that individual braces are in the same cone. Having recognised this problem, its solution is simple, as shown in section 3.4.

If the cantilever is to be braced to $P = 8$, $\mathbf{K}(P) = \mathbf{K} - P\mathbf{S}$ is formed and Choleski factorisation (and inversion) gives

$$\bar{\mathbf{L}}^{-\mathrm{T}} = [l_{ij}] = \begin{bmatrix} 0.11411 & -0.44621 & 0.11411 & -0.60403 \\ . & 0.83406 & 0 & 1.5161 \\ . & . & 0.11411 & -0.07304 \\ . & . & . & 0.24170 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{I}} = \lceil 1\,1\,1 -1 \rfloor \text{ i.e. } p = 4.$$

Equation (16) now shows that braces applied to single freedoms 1, 2 or 4 will work, and bracing freedom 3 alone will not. Any brace connecting freedoms 1, 2 and 4 with positive multipliers is also feasible, from convexity, but we must first ensure that the individual braces are in the same cone. As the cones are right angled, we need only check the sign of the scalar product of 2 feasible braces. Freedoms 1 and 2 are in opposite cones, as are 1 and 4, so a brace such as $\boldsymbol{g} = [1\ -1\ 0\ -1]$ is immediately seen as feasible.

At $P = 15$

$$\bar{\mathbf{L}}^{-\mathrm{T}} = \begin{bmatrix} 0.12910 & -0.31277 & 0.12910 & 0.11213 \\ . & 0.83406 & 0 & -0.95403 \\ . & . & 0.12910 & -0.12281 \\ . & . & . & 0.32750 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{I}} = \lceil 1\ -1\ 1\ 1 \rfloor \text{ i.e. } p = 2.$$

With p = 2, bracing higher numbered freedoms 3 or 4, or any combination, will not be effective. From eqn (16), bracing freedom 1 is effective (with a required brace stiffness of 19.26, as before), and bracing freedom 2 is not. Only the one single freedom brace is feasible, so no convex combinations are offered, but other feasible braces are not hard to find—for example the combination of rows 2 and 4 given by $\boldsymbol{g} = [0\ 1\ 0\ 3]$ will clearly satisfy eqn (11).

To investigate connections capable of bracing fully to the second buckling load of $P_2 = 22.946$, we formulate at just below this value, to avoid the singularity. At $P = 22.94$

$$\bar{\mathbf{L}}^{-\mathrm{T}} = \begin{bmatrix} 0.15628 & -0.23622 & 0.15628 & -3.9916 \\ . & 0.83406 & 0 & -18.196 \\ . & . & 0.15628 & 6.8191 \\ . & . & . & 12.863 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{I}} = \lceil 1\ -1\ 1\ 1 \rfloor \text{ i.e. } p = 2$$

(in passing, the single negative term in $\bar{\mathbf{I}}$ confirms that the target of 22.94 *is* below P_2). No brace connecting to a single freedom will work. However, it is not too difficult to identify some that will, and, say, $\mathbf{g} = [1\ -0.22\ 0\ 0]$ is readily seen as able to brace to full potential. At this last load the mapped central brace $\mathbf{g} = [0\ 1\ 0\ 0]\mathbf{L}^{\mathrm{T}}$ is $[0\ 2.2442\ 0\ 3.1748]$, giving another connection to fully brace the cantilever. Others are readily available through like transformation, or convex combination.

Choleski factorisation at a chosen P gives algorithms, and efficient ones, for establishing feasibility. Equation (15d) generates feasible braces using $\mathbf{L}^{\mathrm{T}}$ with any obviously feasible $\boldsymbol{\beta}$, while eqns (11) and (15c) use the inverse $\mathbf{L}^{-\mathrm{T}}$ to assess any proposed brace $\boldsymbol{g}$ (with eqn (16) giving this for the specific case of $\boldsymbol{g}$ connecting to a single freedom). And convexity allows working combinations to be immediately identified. But despite this information, which may be all that is needed, it is still a step short of the complete description of the feasible region given by the eigenspace.

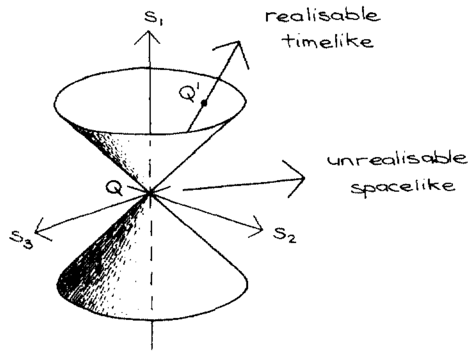

Fig. 6. The world of an event in Minkowski spacetime.

4. ...AND THE SPECIAL THEORY OF RELATIVITY

Einstein's special theory of relativity describes the way that different observers will measure the times and places of events. Its kinematics are based on two postulates:

1. all observers travelling relative to each other at constant velocity will arrive at the same laws of dynamics, and
2. all will agree on a particular velocity.

If the agreed velocity is infinite the result is Galilean kinematics, the basis of Newtonian mechanics, and if finite, it can be defined to be 1, so that the unit of time is that taken to travel the unit of distance at this velocity. The agreed velocity could be anything, but fitting the theory to observations makes it the free-space velocity of light.

Descriptions are often presented in a 4-dimensional spacetime, due to Minkowski. An event in this spacetime is described by a vector **s** of four coordinates, with s_1 the time of the event, and s_2, s_3, s_4 its location in a Euclidean 3-space. Any two events separated by $\Delta\mathbf{s}$ and connected at the unit velocity must have

$$\Delta s^2 = -\Delta s_1^2 + \Delta s_2^2 + \Delta s_3^2 + \Delta s_4^2 = 0 \tag{17}$$

which is clearly agreed by all observers. The separation Δs^2 between *any* two events is also agreed, so that

$$\Delta s^2 = \Delta\mathbf{s}\bar{\mathbf{I}}\Delta\mathbf{s}^{\mathrm{T}} \tag{17a}$$

is invariant, where $\bar{\mathbf{I}} = \ulcorner -1\ 1\ 1\ 1 \lrcorner$. If the measurements of two observers are related by $\mathbf{s}_1 = \mathbf{s}_2\mathbf{A}$ then the transform **A** must satisfy

$$\mathbf{A}\bar{\mathbf{I}}\mathbf{A}^{\mathrm{T}} = \bar{\mathbf{I}}. \tag{18}$$

Matrices **A** are the *Lorentz* transforms which describe how the measurements of one observer will be recorded by another. Some of the well known consequences of these transformations are

1. Time dilation—a clock moving at a velocity v is seen to run slowly by a factor of $\sqrt{1-v^2}$.
2. Spatial contraction—lengths measured parallel to motion are found to be shortened, by the same factor.
3. Bounding velocity—the agreed velocity (of light) is a limit to the velocities of all physical objects. No physical body moving slower than light can be accelerated to a speed faster than light.

Because the velocity of light limits the velocity of signals between observers, the 'world' of an event Q is often drawn in Minkowski spacetime as shown in Fig. 6.

If light is emitted at Q it will travel a distance proportional to travel time, and with the current definition of units, numerically equal to the time s_1. It lies on the upper cone of the diagram. Similarly any light arriving at Q lies on the lower cone. The upper cone and its interior form the *absolute future* of the event Q—a particle at Q could go to any such event Q' in this cone at a speed < 1, and all observers likewise with speeds < 1 would agree that Q' is at a later time than Q. Similarly, the lower cone and its interior form the *absolute past* of Q—the particle could have come from any such point and all the observers mentioned above would agree that this preceded Q. The surface of the lower cone is the *visible now* of Q—a photograph taken at Q would contain a sample of this cone (ignoring the minor complication of a finite shutter speed). Events outside the cones are neither in Q's absolute past or future, and are described as *elsewhere*. Travel at a realisable speed, < 1, is represented by a vector inside the cone, and is *timelike*. Travel at speed > 1 is a vector outside the cone, is *spacelike*, and is unrealisable. In terms of the separation Δs^2 of eqns (17) and (17a), the cone surface contains all vectors with zero separation, the interior timelike vectors have negative separation and the spacelike exterior vectors have positive separation. Rindler (1977) gives a thorough, and recommended, presentation of relativity.

At the risk of pointing out the obvious, the Minkowski description of the kinematics, presented above, is strongly analogous to the bracing problem—eqns (12) and (17a) are identical, as are (13a) and (18), and Figs 5 and 6 are effectively so. The different observers of relativity are the different coordinate systems of the structure. A realisable movement is a feasible brace, and an infeasible brace is elsewhere. Spacetime separations are bracing flexibilities and light particles are braces on the feasible boundary, requiring an unyielding support. And just as all observers travelling slower than light can equally see themselves as at rest in the centre of the universe, all feasible braces are either on the boundary, or can be considered at the centre of the cone. Equation (14) contains all that is essential in the Lorentz transforms, including the factors for time dilation and spatial contraction, and by implication, the relation for the relativity of velocity.

Despite this strong analogy, there are obvious fundamental differences between the two problems. The kinetics of relativity does not seem to have a counterpart, and we don't see the equivalent of mass transformations, or the most renowned of all, $E = mc^2$. But in other ways the bracing problem is the more complex. Relativity is concerned *only* with the Lorentz transformations, but the bracing problem is not—relativity is spatially isotropic to mirror the observed world, but eqn (12) is isotropic purely by choice. In the $\boldsymbol{\alpha}$ space of the eigenvectors, and in many more, including the original space of the freedoms, the cone does not have a circular cross section—it may not even be symmetric, though it has to be convex. One buckling problem will produce relativity and relativity only, but is a very specific one. This structure has only two different buckling loads, a distinct fundamental load, and a multiple second mode. A fairly unusual structure.

And while all feasible braces are equal, some are more equal than others. The brace $\mathbf{g}_1$ can be distinguished as the only one not to suffer from the law of diminishing returns (in the above oddity, $\mathbf{g}_1$ is the only brace to remain feasible as $P \to P_2$—its image is the central brace of Fig. 4). But it is when we look at the loading history of a structure that we see the major difference. A structure starts off as stable, and as the load increases the fundamental load is passed, followed by the second and subsequent buckling loads. With 'stable directions' spacelike and 'buckled directions' timelike, the structure starts off in a completely spatial, timeless world. At buckling, one of its spatial dimensions is traded for time, giving a metric space with analogies to the real world. But as loading progresses and the second buckling load is passed, a further, similar trade takes place, and we enter a new world with two dimensions of time. The Minkowski-like world of the structure progressively accrues timelike dimensions as it sheds spatial dimensions, until it finally runs out of space (if it does not first run out of positive modes).

5. CLOSING REMARKS

A structure can always be braced from its first buckling load up to its second load using a single rank 1 brace, such as an added support or truss member. This is well known.

And checking whether a given brace will suffice is straight forward. Investigating which braces will produce a given load is less so. An answer has been given in terms of the loads and modes of the original structure, showing that the feasible braces form a convex set which shrinks continually as the required load increases, finally having one dimension less as the second buckling load of the structure is reached. This solution is computationally intensive, needing a complete solution of the buckling eigenvalue problem. A computationally attractive, but less complete answer requires only that the Choleski transformation of the loaded structure be computed.

Transforming from one description to another leaves bracing feasibilities as invariants in a (pseudo)metric space of signature one. This makes the problem analogous to the special theory of relativity, with its description in Minkowski space. It is a surprise to find that two so dissimilar problems have the same mathematical description. With hindsight, of course, there is no surprise. The connection is simply the signature of the metric. Hindsight works a lot better than foresight, but is a lot less interesting.

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